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SUPERSELECTION RULES AND CONSERVATION OF HEAVY PARTICLES.

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Superselection Rules and Conservation of Heavy Particles

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In Sec. I, the superselection rule regarding the parity of spinor-particle number is derived from a consideration which does not depend on the double time-reversal. In Sec. II, liberating the space-parity operator P from the customary assumption $P^2 = 1$, we obtain a more restrictive selection rule than the usual one for space-symmetry. This leads, in Sec. III, to a new selection rule that the difference between the ''particle'' number and the ''antiparticle'' number should conserve. The so-called conservation of heavy particles can be incorporated in this rule. This invariance, combined with the invariance for a rotation about the third axis of the isotopic spin space, results in the law of charge conservation. These three selection rules are suspected to be also superselection rules.

I. Superselection Rule for Parity of Fermion Number

If we can divide the entire Hilbert space into a set of subspaces: $\mathfrak{M}_1, \mathfrak{M}_2, \ldots$ such that

$$(\Psi_i, \Psi_j) = 0, \tag{1.1}$$

$$(\Psi_i, S\Psi_j) = 0, \tag{1.2}$$

with $\quad \Psi_i \in \mathfrak{M}_i, \quad \Psi_j \in \mathfrak{M}_j, \quad i \neq j,$

where S is the S-matrix, then we speak of a selection rule. If, furthermore, the set of subspaces satisfies, besides (1.1) and (1.2),

$$(\Psi_i, Q\Psi_j) = 0 \tag{1.3}$$

for all physically observable quantities Q, then, according to W.W.W.,[1] we can speak of a superselection rule. If a superselection rule holds, then the relative phase-factor $e^{i\alpha}$ in $\Psi = (\Psi_j + e^{i\alpha}\Psi_j)/\sqrt{2}$ is absolutely unobservable.

It is obvious that any hermitian or unitary operator W that commutes with the S-matrix can serve as the generator of a selection rule. Similarly, if there is a hermitian or unitary operator W that commutes not only with S but also with all Q, it will engender a superselection rule. The subspaces $\mathfrak{M}_i$ will be given by

$$\mathfrak{M}_i = \{\Psi_i \mid W\Psi_i = W_i\Psi_i\} \tag{1.4}$$

with $W_i \neq W_j$ for $i \neq j$. Since the S-matrix can be built from the Hamiltonian and the Hamiltonian must be among the Q's, the generator W of a superselection rule can be characterized simply by its commutability with all Q.

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1. G.C. Wick, A.S. Wightman and E.P. Wigner, Phys.Rev., 88, 101 (1952)

Any physical quantity is of the general form:

$$Q = \psi_1^\dagger \psi_2^\dagger \cdots \psi_\mu^\dagger \, O \, \varphi_1 \varphi_2 \cdots \varphi_\nu \, F \quad , \tag{1.5}$$

or a sum of terms of this type. F is some quantity derived from the boson fields, and φ and $\psi^\dagger$ are linear, respectively, in absorption operators and emission operators of spinor fields. It is understood that the spinor fields have already been separated into "particle" fields and "antiparticle" fields in the expression (1.5). Let us call k a common divisor of all possible values of $|\mu - \nu|$ appearing in all physically observable quantities Q. In this definition, $\mu - \nu = 0$ is to be considered as a multiple of any arbitrary integer.

We now introduce a unitary transformation defined by

$$W_k = \prod_i \omega_k^{N_i} = \prod_i \left(1 - N_i + \omega_k N_i \right) , \tag{1.6}$$

where ω_k is any one of the k-th roots of unity, and the running index i is to extend over all spinor eigen-states. N_i is the occupation number operator for the spinor eigen-state i. Then we have

$$Q' = W_k Q W_k^{-1} = \omega_k^{\mu - \nu} Q = Q \, , \tag{1.7}$$

meaning that a state Ψ and its transform $\Psi' = W_k \Psi$ are physically indistinguishable. This W_k certainly engenders a superselection rule. In particular, if we take as k the greatest common divisor of all $|\mu - \nu|$ and put $\omega_k = \exp(2\pi i/k)$, we obtain the most restrictive superselection rule of this type. W_k will then have k different eigenvalues $(\omega_k)^r$ with $r = 0, 1, 2, \ldots k - 1$. This r is nothing but the total number of spinor particles m, modulo k, present in an eigenstate Ψ, for

$$W_k \psi_1^\dagger \psi_2^\dagger \cdots \psi_m^\dagger \Psi_{vacuum} = (\omega_k)^m \psi_1^\dagger \psi_2^\dagger \cdots \psi_m^\dagger \Psi_{vacuum} \, . \tag{1.8}$$

It is quite natural to assume in general that $\mu + \nu$ (therefore $\mu - \nu$) is an even number for any physically observable Q, since physical observation is essentially a "classical" process, and therefore Q must be a tensorial quantity, or a certain combination of tensorial quantities. As a consequence thereof, $k = 2$ is at least a common divisor.[2] W.W.W.'s superselection rule[1] regarding total angular momentum corresponds to this case and can be considered as having general validity. W_2 has two eigen-values ± 1 which gives the parity of the spinor-particle number.

The derivation of the superselection rule given here has the advantage of being independent of the time-reversal consideration[2] and of suggesting the possible generalization. A special spinor field for which not only W_2 but W_k with, say, $k = 4$ also engenders a selection rule (whether or not it is a superselection rule) is quite conceivable. If the heavy particles not only satisfy the so-called law of conservation of heavy particles but also are incapable of pair creation, then W_k with any arbitrary k will engender a selection rule.

II. Generalized Space-Parity Operators for Spinor Fields

Let us write $P\Psi$ for the state into which a state Ψ is transformed by space-inversion ($\mathfrak{r} \to -\mathfrak{r}$). It is usually assumed that P^2 is a c-number, and it is customary to fix the phase-factor by

$$P^2 = 1. \tag{2.1}$$

However, on account of the superselection rule of the last section, Ψ and $W_2\Psi$ are physically indistinguishable. Therefore, we can as well assume P^2 to be a multiple of W_2. Adjusting the phase-factor,

2. See Eqs. (4.25), (8.15) and subsequent discussion in S. Watanabe, Phys. Rev. 84, 1008 (1951).

we may then put

$$P^2 = W_2 \, . \tag{2.2}$$

More generally, we can expect that, besides W_2, there may be other unitary operators W, such that Ψ and $W\Psi$ are physically indistinguishable. Thus, we have to assume, for the most general parity operator, the condition:

$$P^2 = W, \tag{2.3}$$

where W should commute with all physical quantities.

Conversely, we can determine, according to the current field theory, the general expression of P which transforms a state Ψ into its space-inverted state $P\Psi$. Then, $P^2\Psi$ and Ψ must have exactly the same expectation values for all the known physical quantities. Such P^2 will not necessarily be a c-number. From this, we shall be able to discover a W which commutes with all the known physical quantities.

Let $a_{\pm}(\mathbf{k},\mu)$ represent the absorption operator of a spinor particle with linear momentum $\mathbf{k}$ and spin μ, where $\mu = 1$ when the spin is parallel to the propagation direction and $\mu = -1$ when it is antiparallel. The double sign on $a_{\pm}$ designates a "particle" and an "antiparticle". Then the space-symmetry of physical laws is guaranteed by the parity operator:

$$P\, a_{\pm}(\mathbf{k},\mu) P^{-1} = \pm\, e^{\pm i\alpha}\, a_{\pm}(-\mathbf{k},-\mu)\, , \tag{2.4}$$

where α is entirely arbitrary. The spin $-\mu$ referring to $-\mathbf{k}$ has actually the same direction as μ referring to $\mathbf{k}$. Reiteration of (2.4) yields

$$P^2 a_{\pm}(\mathbf{k},\mu)(P^{-1})^2 = e^{\pm 2i\alpha}\, a_{\pm}(\mathbf{k},\mu)\, , \tag{2.5}$$

showing that $\alpha = 0$ or $\pm\pi$ will lead to (2.1), while $\alpha = \pm\pi/2$ will lead to (2.2), for W_2 anticommutes with any emission or absorption operator.

More generally, (2.5) shows that P^2 is given by

$$P^2 = W = e^{-2i\alpha(N_+ - N_-)} , \tag{2.6}$$

with

$$N_+ = \sum_i N_{+i} \quad , \quad N_- = \sum_j N_{-j} \quad , \tag{2.7}$$

where N_{+i} and N_{-j} are the occupation number operators of the "particle" state i and the "antiparticle" state j. The result (2.6) can also be obtained from the explicit expression of the general parity operator P.

W given in (2.6), with arbitrary α, has eigen-values $e^{-2i\alpha r}$ with $r = 0, \pm 1, \pm 2, \ldots \pm \infty$, where r is the total number of "particles" minus the total number of "antiparticles" present in the corresponding eigen-state.

The connection between the eigen-value π' of the generalized parity operator P' and the eigen-value π of the ordinary parity operator P (2.1) can also easily be seen from (2.4):

$$P'\Psi = \pi'\Psi \quad , \quad P\Psi = \pi\Psi , \tag{2.8}$$

with

$$\pi' = e^{-i\alpha(N_+ - N_-)} \pi , \tag{2.9}$$

the parity of the vacuum-state serving as the standard: $\pi = \pi^0$, say, $= +1$.

Since the selection rule engendered by the ordinary P as well as the one engendered by the generalized P' must be observed, the actual restriction on transitions is much more stringent than usually assumed. Namely, transition between two states are prohibited not only when the ordinary parity values π are different in the two states, but also when the values of $N_+ - N_-$ are different in the two states. Only for the

states for which $N_+ - N_- = 0$, π' becomes equal to π. Such a state may be called a state with definite parity, and a state for which $N_+ - N_- \neq 0$ may be called a state with indeterminate parity. A pure boson state has obviously a definite parity.[3] A corrolary of the above stated selection rule is that a transition between a state with definite parity and a state with indeterminate parity is prohibited.

For instance, a state represented by

$$\Psi = \frac{1}{\sqrt{2}} \left[\bar{a}_+(\mathbf{k},1)\, \bar{a}_+(-\mathbf{k},1) + \bar{a}_+(-\mathbf{k},-1)\, \bar{a}_+(\mathbf{k},-1) \right] \Psi_{vacuum} \qquad (2.10)$$

has the value $\pi = 1$ for the ordinary parity operator (2.1) while it has $\pi' = -1$ for the parity operator satisfying (2.2). As a consequence the state (2.10) cannot pass, for instance, to a state of two positive pions. Such a transition would be allowed by the usually known selection rules and even by the superselection rule of the last section, for the number of spinor particles involved is even in either state.

III. Conservation of Heavy Particles and Charge Conservation.

The selection rule engendered by W given in (2.6) means that the number of spinor particles can change only by pairs of a "particle" and an "antiparticle", not necessarily of the same field.[4] The selection rule engendered by the generalized parity operator is nothing but the combination of this rule and the ordinary parity rule. The selection rule in question may be referred to, for simplicity, as the pair-

3. These statements are true insofar as the generalized parity operator satisfying (2.6) is concerned. For the generalization of the parity operator for the charged bosons, see the remark at the end of this paper.
4. Of course, a simultaneous creation of a "particle" and annihilation of another "particle", or a simultaneous creation of an "antiparticle" and annihilation of another "antiparticle" is allowed.

creation rule, where the word pair does not necessarily mean a pair belonging to the same field.[5]

If we write W of (2.6) explicitly for charged fields and neutral fields, it becomes

$$W = \prod_{ij\ell m} \omega^{N+i} \omega^{*N-j} \omega^{N+o\ell} \omega^{*N-om} , \quad |\omega| = 1 , \tag{3.1}$$

where N_{+i}, N_{-j}, $N_{+o\ell}$ and N_{-om} respectively are the occupation number operators of positive spinor state i, negative spinor state j, neutral "particle" state ℓ and neutral "antiparticle" state m. The corresponding conservation law is

$$N_{+} - N_{-} + N_{+o} - N_{-o} = \text{invariant}. \tag{3.2}$$

If we identify neutron and neutrino (defined as the particle emitted simultaneously with a negative electron) as neutral "particles", then we can see that the pair-creation rule has the effect of excluding undesired transitions[6] in the nucleon-pion and nucleon-lepton interactions. (Conservation of heavy particles). We have to assume the pair-creation rule separately for the nucleon family and the lepton family to exclude some of the undesired transitions.

It is well-known that the nucleon-pion interaction in the symmetric theory has the rotational symmetry in the isotopic spin space. In particular, the unitary operator generating a rotation by angle φ about the third axis of the isotopic spin space is

$$W = \prod_{ij\ell mpq} \omega^{N+i} \omega^{*N-j} \omega^{*N+o\ell} \omega^{N-om} \omega^{2M+p} \omega^{*2M-q} , \tag{3.3}$$

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5. Once discovered, this law can easily be proved without using the double space-inversion simply by showing commutability of W (3.1) with the S-matrix. The derivation given in Sec. II may however be credited for its heuristic value.

6. C.N. Yang & J.Tiomno, Phys. Rev., 79, 495 1950 .

with $\omega = e^{i\varphi/2}$, where M_{+p} and M_{-q} are the occupation number operators of positive boson (pion) state p and negative boson (pion) state q. This unitary transformation yields the conservation law:[7]

$$N_+ - N_- - N_{+o} + N_{-o} + 2M_+ - 2M_- = \text{invariant} . \tag{3.4}$$

By combining (3.2) and (3.4), we obtain the law of conservation of charges:

$$N_+ - N_- + M_+ - M_- = \text{invariant} . \tag{3.5}$$

which is engendered by the unitary transformation:

$$W = \prod_{ijpq} \omega^{N+i} \, \omega^{*N-j} \, \omega^{M+p} \, \omega^{*M-q} . \tag{3.6}$$

Since the three unitary operators (3.1), (3.3), (3.6) not only commute with the S-matrix but also with all the known physical quantities, it seems well grounded to believe that they are generators not only of selection rules but also of superselection rules.

It may be noted that W_2 of Sec. I and the three W's of this Sec. III are connected with four special types of simple gauge transformations. W_2 corresponds to the transformation: $\psi \rightarrow -\psi$ for all spinors. (3.1) corresponds to $\psi_\pm \rightarrow e^{\pm i\alpha}\psi_\pm$ for charged as well as neutral spinors where the double sign stands for "particle" and "antiparticle". (3.3) has the counterpart: $\psi_\pm \rightarrow e^{\pm i\alpha}\psi_\pm$ for charged spinors, $\psi_\pm \rightarrow e^{\mp i\alpha}\psi_\pm$ for neutral spinors and $u_\pm \rightarrow e^{\pm 2i\alpha} u_\pm$ for charged bosons. (3.6) corresponds to $\psi_\pm \rightarrow e^{\pm i\alpha}\psi_\pm$ for charged spinors and

7. We have to include both nucleons and leptons in the N's to make this law valid also for nucleon-lepton interaction. This law alone does not exclude all the undesired transitions.

$u_{\pm} \rightarrow e^{\pm i\alpha} u_{\pm}$ for charged bosons.

The method of generalization of parity operator used in Sec. II with regard to spinor fields can be extended to the charged boson fields. However, in this case, the unitary transformation corresponding to (2.6) cannot be written exclusively with regard to the charged boson fields alone, and has to be linked to the spinor fields. As a result, we get a unitary transformation of the type (3.3), which was introduced in this paper from a different line of consideration.

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